LEVEL-CROSSINGS OF SYMMETRIC RANDOM WALKS AND THEIR APPLICATION

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ABSTRACT. Let X_1, X_2, ... be a sequence of independently and identically distributed random variables with $\mathsf{E}X_1 = 0$, and let $S_0 = 0$ and $S_t = S_{t-1} + X_t$, $t = 1, 2, \ldots$, be a random walk. Denote $\tau = \begin{cases} \inf\{t > 1 : S_t \leq 0\}, & \text{if } X_1 > 0, \\ 1, & \text{otherwise.} \end{cases}$ Let α denote a positive number, and let L_α denote the number of level-crossings from the below (or above) across the level α during the interval $[0, \tau]$. Under quite general assumption, an inequality for the expected number of level-crossings is established. Under some special assumptions, it is proved that there exists an infinitely increasing sequence α_n such that the equality $\mathsf{E}L_{\alpha_n} = c\mathsf{P}\{X_1 > 0\}$ is satisfied, where c is a specified constant that does not depend on n. The result is illustrated for a number of special random walks. We also give non-trivial examples from queuing theory where the results of this theory are applied.

1. INTRODUCTION

In this article, we discuss beautiful properties of symmetric random walks. All random walks considered in this article are assumed to be one-dimensional.

A symmetric random walk is a well-known object in probability theory, and there are many classic books such as [8], [16] that give a very detailed its study. Nevertheless, even at elementary level, symmetric random walks are very appealing and have astonishing properties of their level-crossings.

Let X_1, X_2, ... be a sequence of independent random variables taking the values $+1$ and -1 each with probability $\frac{1}{2}$. The *simplest symmetric random walk* is defined as $S_0 = 0$, and $S_n = X_1 + X_2 + \ldots + X_n$. Let $\tau = \inf\{i > 0 : S_i = 0\}$ be a stopping time. For integer positive α, let L_α denote the total number of events $\{X_t = \alpha - 1$

and $X_{t+1} = \alpha$} that occurs during the interval $[0, \tau]$, i.e. for $t = 0, 1, \ldots, \tau - 1$. The remarkable property of this random walk is that $\mathsf{E}L_\alpha = \frac{1}{2}$ for all α.

In a book [17], this property is classified as a paradox in probability theory. The proof of this level-crossing property (in slightly different formulation) can be found in [18], p. 411, where that proof is a part of a special theory and seems to be complicated. The level-crossing properties of symmetric random walks are very important in many applications of Applied Probability. For instance, the aforementioned level-crossing property for symmetric random walk is directly reformulated in terms of the $M/M/1/n$ queuing system. If the expectation of interarrival and service times in that queuing system are equal, then the expected number of losses during a busy period is equal to 1 for all $n \geq 0$. Surprisingly, this property holds true for the more general $M/GI/1/n$ queuing system as well (see [1], [12], [19] and a survey paper [3] for further information).

In the present paper, we study properties of level-crossings for general symmetric random walks, which are defined as $S_0 = 0$ and $S_n = X_1 + X_2 + \ldots + X_n$, where X_1, X_2, $\ldots$ are independently and identically distributed random variables with $\mathsf{E}X_1 = 0$. The exact formulation of the problem and relevant definitions are given later.

There is a huge number of papers, where the level-crossings are used and serve as a main tool of analysis. We refer only a few papers that have a theoretical contribution in the areas. Special questions on asymptotic behavior of crossings moving boundaries have been studied in [11] (see also [14], p. 536 as well as the references in [11] about previous studies). The asymptotic number of crossings that are required to reach a high boundary has been studied in [15]. Asymptotic behavior of random walks with application to statistical theory has been studied in [7]. Level-crossings for Gaussian random fields have been studied in [10] and [4] and those for Markov and stationary processes in [5] and [6].

The present paper addresses an open question related to general symmetric random walks: *whether or not the aforemention property of symmetric random walk is valid for general random walk?* Having a negative answer on this question under the general setting, in the paper we find the conditions under which the aforementioned result on the simplest symmetric random walk can be extended to more

general symmetric random walks, i.e. conditions when the expected number of level-crossings remains unchanged when the level α varies.

The article is organized as follows. In short Section 2 we classify symmetric random walks, which is useful for further presentation of the results. In Section 3 we give necessary definitions, examples and specifically a counterexample showing that the aforementioned property of level-crossings being correct for the simplest symmetric random walk is no longer valid for general symmetric random walks. In Section 4 we prove the main results of this paper on level-crossings in general symmetric random walks. In Section 5 we discuss a nontrivial application of these results in queuing theory. In Section 6 we conclude the paper.

2. CLASSIFICATION OF SYMMETRIC RANDOM WALKS

Let X_1, X_2,... be a sequence of independently and identically distributed random variables, and let $S_0 = 0$ and $S_{t+1} = S_t + X_{t+1}$, $t = 0, 1 \ldots$ The sequence $\{S_t\}$ is called *random walk*. A random walk is called *E-symmetric* if $\mathsf{E}X_1 = 0$. If, in addition, $\mathsf{P}\{X_1 > 0\} = \mathsf{P}\{X_1 < 0\}$, then the random walk is called *P-symmetric*. A random walk is called *purely symmetric* if for any x, $\mathsf{P}\{X_1 \leq x\} = \mathsf{P}\{X_1 \geq -x\}$, and $\mathsf{E}|X_1| < \infty$.

Apparently, any purely symmetric random walk is P-symmetric, and any P-symmetric random walk is E-symmetric, i.e.

$$\text{Purely symmetric RW} \Longrightarrow P\text{-Symmetric RW} \Longrightarrow E\text{-Symmetric RW},$$

In the case where X_1 takes the values $+1$ or -1 with the equal probability $\frac{1}{2}$, the random walk is called *simplest symmetric random walk*.

3. DEFINITIONS, EXAMPLES AND COUNTEREXAMPLES

Definition 3.1. The stopping time for the random walks is as follows:

$$\tau = \begin{cases} \inf\{t > 1 : S_t \leq 0\}, & \text{if } X_1 > 0, \\ 1, & \text{if } X_1 \leq 0. \end{cases}$$

Definition 3.2. For any positive α, by the number of level-crossings across the level α we mean the total number of events $\{S_{t-1} < \alpha \text{ and } S_t \geq \alpha\}$, where the

index t runs the integer values from 1 to τ. The number of level-crossings across
the level α is denoted L_α.

We start from the elementary example for the following purely symmetric random
walk.

Example 3.3. Let X_1 take values $\{-1, -2, +1, +2\}$ each with the probability $\frac{1}{4}$.
Taking the level $\alpha = 1$ it is easy to see that the expected number of level-crossings
is equal to $\frac{1}{2}$ exactly. Indeed, there is probability $\frac{1}{2}$ that X_1 is negative and $\frac{1}{2}$ that
it is positive. So, if $X_1 > 0$, then the value S_1 is not smaller than 1. The level 1 is
once reached immediately, and the counter of level-crossings is set to 1. After this,
the excursion of the random walk will be always above the point 1 until the time
$\tau - 1$. During the time interval $[1, \tau - 1]$ this point can be reached from the above
only, but not from the below. Finally, until the stopping time τ, the level $\alpha = 1$ is
no longer reached or intersected from the below. In this case, the total expectation
formula gives $\mathsf{E}L_1 = \frac{1}{2}$. As we see, this case is in agreement with the result in the
case of the simplest symmetric random walk, where $\mathsf{E}L_n = \frac{1}{2}$ for all $n \geq 1$.

Thus, in Example 3.3 we obtain $\mathsf{E}L_1 = \frac{1}{2}$. Is it true that $\mathsf{E}L_\alpha = \frac{1}{2}$ for all positive
α as well? Unfortunately, by direct calculations it is hard to check this property
even for $\alpha = 2$. We leave this question now, but answer it later.

Example 3.4. Consider another example, related now to a P-symmetric random
walk. Assume that X_1 takes the value 2 with probability $\frac{1}{2}$, the value -1 with
probability $\frac{1}{4}$ and the value -3 with probability $\frac{1}{4}$. Apparently, in the case $\alpha = 1$
the expected number of level-crossings from the below across this level is equal to
$\frac{1}{2}$. As in the example above, the level 1 is intersected in the first step of the random
walk (if X_1 is positive), and the following excursion is always above this level before
the time $\tau - 1$. Using the total expectation formula, as in the case of Example 3.3,
we obtain $\mathsf{E}L_1 = \frac{1}{2}$. For the same random walk, assume now that $\alpha = 2$. If X_1
is positive, then the level 2 is reached immediately. However, there is the positive
probability that the random walk will return to the level 1 and then intersect the
level 2 once again. Hence, $\mathsf{E}L_2 > \frac{1}{2}$.

Thus, $\mathsf{E}L_\alpha$ depends on α in general. Following this, in the present paper there
are considered two main questions associated with the behaviour of $\mathsf{E}L_\alpha$ when α

varies. First, *under what conditions* $\mathsf{E}L_\alpha$ *remains the same when* α *varies?* Second, *is* $\mathsf{P}\{X_1 > 0\}$ *the minimum of all possible values of* $\mathsf{E}L_\alpha$? *For what family of symmetric random walks the last is true?*

4. LEVEL-CROSSINGS OF E-, P- AND PURELY SYMMETRIC RANDOM WALKS

In this section we establish the properties of the level-crossings for P-, E- and purely symmetric random walks, and thus answer on the questions formulated in Section 3.

According to well-known results in probability theory (such as the second Borel-Cantelli lemma and Markov property, for instance), it is easy to conclude that any E-symmetric random walk is recurrent in the sense that $\mathsf{P}\{\tau < \infty\} = 1$.

In the following we use the following notation. A random variable X_t, $t = 1, 2, \ldots$, is represented

$$X_t = \begin{cases} X_t^+, & \text{if } X_t > 0, \\ X_t^-, & \text{if } X_t \leq 0, \end{cases}$$

where X_t^+ takes the only positive values of X_t, while X_t^- takes the nonpositive values of X_t.

The sequences $\{X_t^+\}$ and $\{X_t^-\}$, $t = 1, 2, \ldots$ are independent and consist of independently and identically distributed random variables with the expectations $\frac{a}{\mathsf{P}\{X_t > 0\}}$ and $\frac{a}{\mathsf{P}\{X_t \leq 0\}}$, respectively. As well, we denote $S_0^+ = S_0^- = 0$ and correspondingly $S_t^+ = S_{t-1}^+ + X_t \mathsf{I}\{X_t > 0\}$ and $S_t^- = S_{t-1}^- - X_t \mathsf{I}\{X_t \leq 0\}$, so $S_t = S_t^+ - S_t^-$.

Denote by $t_1(\alpha)$ the first time during the time interval $[0, \tau]$ (if any) such that $S_{t_1(\alpha)} \geq \alpha$, and by $\tau_1(\alpha)$ denote the first time after $t_1(\alpha)$ such that $S_{\tau_1(\alpha)} < \alpha$. Note, that existence of the time $\tau_1(\alpha)$ is associated with the existence of the time $t_1(\alpha)$. If level α is not reached in the interval $[0, \tau]$, then the time $\tau_1(\alpha)$ does not exist either. Next, let $t_2(\alpha)$ be the first time after $\tau_1(\alpha)$ and during the time interval $[0, \tau]$ such that $S_{t_2(\alpha)} \geq \alpha$, and let $\tau_2(\alpha)$ be the first time after $t_2(\alpha)$ such that $S_{\tau_2(\alpha)} < \alpha$. As above, the existence of $t_2(\alpha)$ is associated with that of $t_1(\alpha)$, and, in turn, the existence of $\tau_2(\alpha)$ is associated with that of $t_2(\alpha)$. The times $t_i(\alpha)$ and $\tau_i(\alpha)$ $(i > 2)$ are defined similarly.

Note, that the existence of $t_2(\alpha)$ and, respectively, $\tau_2(\alpha)$ generally depend on α as well. In Examples 3.3 and 3.4 for the specific level $\alpha = 1$ the second level-crossing does not exist. So, in the following we reckon that existence of $t_i(\alpha)$ and $\tau_i(\alpha)$ for $i = 2$ (and hence for $i \geq 2$) are guaranteed with choice of level α, that is, the process is assumed to be defined in the probability space $\{\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_\alpha)_{\alpha \geq \alpha_0}, \mathsf{P}\}$ with an increasing family of filtrations $\mathcal{F}_\alpha$ such that $\mathfrak{A}_2 \in \mathcal{F}_{\alpha_0}$, and $\mathsf{P}\{\mathfrak{A}_2\} > 0$.

Assumption 4.1. Let $\mathfrak{A}_i(\alpha)$ denote the event "the ith crossing of the level α occurs during the time interval $[0, \tau]$", and assume that $\mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1(\alpha)\} > 0$, $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} > 0$,

$$(4.1) \qquad\qquad \mathsf{E}\{S_{t_2(\alpha)-1}|\mathfrak{A}_2(\alpha)\} = \mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1(\alpha)\},$$

and

$$(4.2) \qquad\qquad \mathsf{E}\{S_{\tau_2(\alpha)}|\mathfrak{A}_2(\alpha)\} = \mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\}.$$

Remark 4.2. In general, it is hard to check Conditions (4.1) and (4.2). However, in some cases conditions (4.1) and (4.2) are satisfied automatically. This is demonstrated in the two examples given below.

Example 4.3. Consider the following E-symmetric random walk. Assume that X_t takes values 1 with probability $\frac{2}{3}$ and -2 with probability $\frac{1}{3}$. Take $\alpha \geq 2$, and assume for convenience that α is integer. Then, it is readily seen that under an occurrence of the event $\mathfrak{A}_i(\alpha)$ ($i = 1, 2, \ldots$), we always have $S_{t_i(\alpha)-1} = \alpha - 1$. Moreover, $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} > 0$, and $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} = \mathsf{E}\{S_{\tau_2(\alpha)}|\mathfrak{A}_2(\alpha)\}$.

Example 4.4. Assume that X_t^+ takes a single positive value 1 and $(-X_t^-)$ is a geometrically distributed random variable with mean m. Assume that $\mathsf{P}\{X_t > 0\} = \frac{m}{1+m}$. Then, $\mathsf{E}X_t = 0$, and it is the convention. Apparently, that for any given integer level $\alpha \geq \alpha_0$ the event $\mathfrak{A}_2(\alpha)$ occurs, than $S_{t_1(\alpha)-1}$ and $S_{t_2(\alpha)-1}$ both are equal to $\alpha - 1$. In addition, $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} = \mathsf{E}\{S_{\tau_2(\alpha)}|\mathfrak{A}_2(\alpha)\}$. The value α_0 is assumed to be chosen such that both $\mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1(\alpha)\} > 0$ and $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} > 0$. That is we set $\alpha_0 = \max[1, -\mathsf{E}\{S_\tau|X_1 > 0\}]$.

Examples 4.3 and 4.4 fall into the category of a special class of random walks where X_t^+ takes only one positive value d, while X_t^- takes the values $\{0, -d, -2d, \ldots\}$, some of them can have probability 0. This class is considered later by Theorem 4.9.

The next example demonstrates the case where Assumption 4.1 is not satisfied.

Example 4.5. Consider a symmetric random walk where X_t^+ takes a single positive value 1, while $(-X_t^-)$ is an exponentially distributed random variable with mean 1. According to the well-known property of exponential distribution, $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} = \mathsf{E}\{S_{\tau_2(\alpha)}|\mathfrak{A}_2(\alpha)\} = \alpha - 1$ (α is assumed to be greater than 1). However, it is readily seen that generally $\mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1(\alpha)\} \neq \mathsf{E}\{S_{t_2(\alpha)-1}|\mathfrak{A}_2(\alpha)\}$. Indeed, let α be integer, say 2. Then, clearly $\mathsf{P}\{S_{t_1(2)-1} = 1|\mathfrak{A}_1(2)\} > 0$, while $\mathsf{P}\{S_{t_2(2)-1} = 1|\mathfrak{A}_2(2)\} = 0$. Hence, it is easy to find that $\mathsf{E}\{S_{t_2(2)-1}|\mathfrak{A}_2(2)\} = \frac{3}{2}$, while $\mathsf{E}\{S_{t_1(2)-1}|\mathfrak{A}_1(2)\} < \frac{3}{2}$.

Theorem 4.6. *Let S_t be a E-symmetric random walk, and let Assumption 4.1 be satisfied for some $\alpha > 0$. Then, there exists an infinitely increasing sequence of levels α_n such that*

$$(4.3) \qquad \mathsf{E}L_{\alpha_n} = \mathsf{E}L_\alpha = \frac{a - \mathsf{E}\{S_\tau|X_1 > 0\}}{a - b}\mathsf{P}\{X_1 > 0\},$$

where $a = \mathsf{E}\{X_1|X_1 > 0\}$ and $b = \mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1(\alpha)\} - \mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\}$.

Proof. According to the total expectation formula

$$\mathsf{E}L_\alpha = \mathsf{E}\{L_\alpha|X_1 > 0\}\mathsf{P}\{X_1 > 0\} + \mathsf{E}\{L_\alpha|X_1 \leq 0\}\mathsf{P}\{X_1 \leq 0\}$$

$$= \mathsf{E}\{L_\alpha|X_1 > 0\}\mathsf{P}\{X_1 > 0\}.$$

Hence, the challenge is to prove first that $\mathsf{E}\{L_\alpha|X_1 > 0\} = \frac{a-\mathsf{E}\{S_\tau|X_1>0\}}{a-b}$ for a given $\alpha > 0$, and then to build an increasing sequence of α_n: $\alpha < \alpha_1 < \alpha_2 < \ldots$ where $\mathsf{E}L_{\alpha_n} = \mathsf{E}L_\alpha$.

Let $t_1, t_2, \ldots, t_{L_\alpha}$ be a set of times where the events $\{S_{t_i-1} < \alpha$ and $S_{t_i} \geq \alpha\}$ occur, and, respectively, let $\tau_1, \tau_2, \ldots, \tau_{L_\alpha}$ be a set of times where $\{S_{\tau_i-1} \geq \alpha$ and $S_{\tau_i} < \alpha\}$. The notation for $t_1, t_2, \ldots$ and $\tau_1, \tau_2, \ldots$ is similar to the above, but the parameter α is omitted from the notation for the sake of convenience.

Taking into consideration that the sequence $\{S_t\}$ is a martingale and Assumption 4.1 is satisfied, we have the following properties:

$$(4.4) \qquad \mathsf{E}\{S_{t_i-1}|\mathfrak{A}_i\} = \mathsf{E}\{S_{t_j-1}|\mathfrak{A}_j\},$$

and

(4.5) $$\mathsf{E}\{S_{\tau_i}|\mathfrak{A}_i\} = \mathsf{E}\{S_{\tau_j}|\mathfrak{A}_j\},$$

where $i, j = 1, 2, \ldots, t_{L_\alpha}$.

Note, that the expectations that defined by (4.4) and (4.5) need not be positive in general, because the random value S_τ is non-positive, and $\mathsf{E}\{S_\tau|X_1 > 0\} \leq 0$. So, Assumption 4.1, where these expectations are assumed to be positive, implies the choice of α for which this assumption is satisfied. As well, $S_{\tau-1} > 0$, $X_\tau < 0$ with probability 1. Then we obtain

$$\mathsf{E}\{S_\tau|X_1 > 0\} = \mathsf{E}\{S_{\tau-1}|X_1 > 0\} + \mathsf{E}\{X_\tau|X_1 > 0\}.$$

Hence, for $\alpha \geq -\mathsf{E}\{S_\tau|X_1 > 0\}$ the expectations that defined by (4.4) and (4.5) are guaranteed to be positive.

These properties enable us to establish easily level-crossing properties of any E-symmetric random walk where Assumption 4.1 is satisfied. Let us scale the original time interval $[0, \tau]$ by deleting the time intervals $[t_i - 1, \tau_i)$, $i = 1, 2, \ldots, L_\alpha$ and merging the corresponding ends. As it done, S_{t_i-1} and S_{τ_i} take distinct values in general, and the difference between their expectations in these ends is denoted to be equal to b, i.e. $b = \mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1\} - \mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1\}$.

Let χ denote the length of remaining intervals partitioned on $L_\alpha + 1$ parts, so this remaining time interval is represented as

$$[0, \chi) = \bigcup_{j=1}^{L_\alpha+1} I_j,$$

where

$$I_1 = [1, t_1 - 1), I_2 = [\tau_1, t_2 - 1), I_3 = [\tau_2, t_3 - 1), \ldots,$$

$$I_{L_\alpha} = [\tau_{L_\alpha-1}, t_{L_\alpha} - 1), I_{L_\alpha+1} = [\tau_{L_\alpha}, \tau).$$

Let η_1^-, η_2^-, ..., $\eta_{L_\alpha+1}^-$ denote the numbers of random variables X_i in the corresponding time intervals I_1, I_2, ..., $I_{L_\alpha+1}$ that take non-positive value, and, respectively, let η_1^+, η_2^+, ..., $\eta_{L_\alpha+1}^+$ denote the numbers of random variables X_i in the corresponding time intervals I_1, I_2, ..., $I_{L_\alpha+1}$ that take positive value. Next, set $\eta^- := \eta_1^- + \ldots + \eta_{L_\alpha+1}^-$, and, respectively, $\eta^+ := \eta_1^+ + \ldots + \eta_{L_\alpha+1}^+$.

Assume that the original random variables X_t^+ and X_t^- all are renumbered after the above time scale procedure and follow in the ordinary order. Then $S_{\eta^-}^-$ and $S_{\eta^+}^+$ can be written as

$$(4.6) \qquad S_{\eta^-}^- = S_{\eta_1^- + \eta_2^- + \ldots + \eta_{L_\alpha+1}^-}^-,$$

and

$$(4.7) \qquad S_{\eta^+}^+ = S_{1 + \eta_1^+ + \eta_2^+ + \ldots + \eta_{L_\alpha+1}^+}^+.$$

The random variable η^+ includes a positive random variable X_1 that starts a random walk in the interval I_1. (Recall that our convention was $X_1 > 0$ and we are going to prove that $\mathsf{E}\{L_\alpha | X_1 > 0\} = \frac{a - \mathsf{E}\{S_\tau | X_1 > 0\}}{a - b}$.) For this reason there is the difference in the notation for $S_{\eta^+}^+$ in (4.7) compared to that for $S_{\eta^-}^-$ in (4.6). There is extra 1 in the subscript line of the right-hand side of (4.7).

As well, there is the difference between the initial value of the random walk and the moment of stopping S_τ. This difference is equal to $-S_\tau$, and its expected value is $-\mathsf{E}\{S_\tau | X_1 > 0\}$. Hence,

$$(4.8) \qquad \mathsf{E}\{S_{\eta^+}^+ - S_{\eta^-}^- | X_1 > 0\} = a + b\mathsf{E}\{L_\alpha | X_1 > 0\} - \mathsf{E}\{S_\tau | X_1 > 0\},$$

where the second term on the right-hand side of (4.8), $b\mathsf{E}\{L_\alpha | X_1 > 0\}$, is calculated due to Wald's identity (e.g. Feller [9], p.384).

Applying Wald's identity once again, we obtain

$$\mathsf{E}\{L_\alpha | X_1 > 0\}\mathsf{P}\{X_t > 0\}\mathsf{E}X_t^+ = a + b\mathsf{E}\{L_\alpha | X_1 > 0\} - \mathsf{E}\{S_\tau | X_1 > 0\},$$

and hence, due to the fact that $\mathsf{E}X_1^+ = \frac{a}{\mathsf{P}\{X_t > 0\}}$ we arrive at

$$\mathsf{E}\{L_\alpha | X_1 > 0\} = \frac{a - \mathsf{E}\{S_\tau | X_1 > 0\}}{a - b}.$$

The first part of the theorem is proved.

Let us now prove that there exists an infinitely increasing sequence of values α_1, α_2, . . . such that Assumption 1 is satisfied for these values as well, and moreover, for all $n = 1, 2, \ldots$

$$\mathsf{E}\{S_{t_1(\alpha_n)-1} | \mathfrak{A}_1(\alpha_n)\} - \mathsf{E}\{S_{\tau_1(\alpha_n)} | \mathfrak{A}_1(\alpha_n)\}$$

$$= \mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\} - \mathsf{E}\{S_{\tau_1(\alpha)} | \mathfrak{A}_1(\alpha)\}$$

$$:= b.$$

Assuming that the event $\mathfrak{A}_1$ occurs, denote by $\mathcal{S}_\alpha$ the set of values $S_{t_1(\alpha)-1}$, denote $\tau(x) = \inf\{t > t_1(\alpha) : S_t \leq S_{t_1(\alpha)-1} | S_{t_1(\alpha)-1} = x\}$, and denote by $\mathcal{Z}_\alpha$ the set of all values of $S_{\tau(x)}$. Then x is an initial (non-random) point of a new random walk and $\tau(x)$ is a random stopping time, and during the time interval $[t_1(\alpha) - 1, \tau(x)]$ the behavior of this random walk is the same as that of the original random walk that starts at zero. Hence,

$$(4.9) \qquad \mathsf{E}\{S_{t_1(\alpha+x)-1} | S_{t_1(\alpha)-1} = x, \mathfrak{A}_1(\alpha + x)\} = \mathsf{E}\{S_{t_1(\alpha)-1} + x | \mathfrak{A}_1(\alpha + x)\},$$

for all possible values $x \in \mathcal{S}_\alpha$.

By the total expectation formula, we obtain:

$$\mathsf{E}\{S_{t_1(\alpha+\mathsf{E}S_{t_1(\alpha)-1})-1} | \mathfrak{A}_1(\alpha + \mathsf{E}S_{t_1(\alpha)-1})\} = 2\mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\}.$$

Hence, $\mathsf{E}\{S_{t_1(\alpha_1)-1} | \mathfrak{A}_1(\alpha_1)\} = 2\mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\}$ for $\alpha_1 = \alpha + \mathsf{E}S_{t_1(\alpha)-1}$. For the level α_1, the same properties as those for the original level α are satisfied. That is,

$$(4.10) \qquad\qquad \mathsf{E}\{S_{t_2(\alpha_1)-1} | \mathfrak{A}_2(\alpha_1)\} = \mathsf{E}\{S_{t_1(\alpha_1)-1} | \mathfrak{A}_1(\alpha_1)\}.$$

Respectively, with the coupling arguments we obtain

$$\mathsf{E}\{S_{\tau_2(\alpha_1)} | \mathfrak{A}_2(\alpha_1)\} = \mathsf{E}\{S_{\tau_1(\alpha_1)} | \mathfrak{A}_1(\alpha_1)\}.$$

Similar arguments of the induction enable us to obtain the relation

$$\alpha_{i+1} = \alpha_i + \mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\},$$

where all required properties of the stopping times are satisfied. This completes the proof of the second part of the theorem as well, and totally completes the proof of the theorem. $\qquad\qquad\square$

Remark 4.7. Assumptions $\mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\} > 0$ and $\mathsf{E}\{S_{\tau_1(\alpha)} | \mathfrak{A}_1(\alpha)\} > 0$ are important. If at least one of them is not satisfied, then the expected number of level-crossings need not be equal to the value that obtained in the statement of the theorem, because in this case, equality (4.8) is not valid, since the level 0 is "overlapped" by negative value(s) in (4.4) or/and (4.5), and one cannot use the value $\mathsf{E}\{S_\tau | X_1 > 0\}$ in expression (4.8).

The result for P-symmetric random walk follows from Theorem 4.6 as a corollary.

Corollary 4.8. *Let S_t be a P-symmetric random walk, and let Assumption 4.1 be satisfied for some $\alpha > 0$. Then, there exists an infinitely increasing sequence of levels α_n such that*

$$\mathsf{E}L_{\alpha_n} = \mathsf{E}L_\alpha = \frac{1}{2} \cdot \frac{a - \mathsf{E}\{S_\tau | X_1 > 0\}}{a - b} \mathsf{P}\{X_1 \neq 0\}.$$

where $a = \mathsf{E}\{X_1 | X_1 > 0\}$, and $b = \mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\} - \mathsf{E}\{S_{\tau_1(\alpha)} | \mathfrak{A}_1(\alpha)\}$.

Proof. Indeed, the relation $\mathsf{P}\{X_1 > 0\} = \mathsf{P}\{X_1 < 0\}$ enables us to conclude that $\mathsf{P}\{X_1 > 0\} = \frac{1}{2}\mathsf{P}\{X_1 \neq 0\}$. Hence, the result follows as a reformulation of Theorem 4.6. $\square$

In the particular case where $\mathsf{P}\{X_1 = 0\} = 0$, for P-symmetric random walks satisfying Assumption 4.1 we have $\mathsf{E}L_{\alpha_n} = \frac{1}{2} \cdot \frac{a - \mathsf{E}\{S_\tau | X_1 > 0\}}{a - b}$ for all values α_n defined in the proof of Theorem 4.6.

The following theorem demonstrates application of Theorem 4.6 to a special type of E-symmetric random walks.

Theorem 4.9. *Suppose that X_t^+ takes only a single positive value d, while X_t^- takes the values $0, -d, -2d, \ldots$ (some of them can have probability 0). Then, for all $\alpha > \max[d, -\mathsf{E}\{S_\tau | X_1 > 0\}]$ we have*

$$\mathsf{E}L_\alpha = \frac{d - \mathsf{E}\{S_\tau | X_1 > 0\}}{d - \mathsf{E}\{S_\tau | X_1 > 0\})\mathsf{P}\{X_1 > 0\}} \mathsf{P}\{X_1 > 0\}.$$

Proof. Indeed, it is readily seen that for $\alpha > \max[d, -\mathsf{E}\{S_\tau | X_1 > 0\}]$ Assumption 4.1 is satisfied, since if the event $\mathfrak{A}_i(\alpha)$ occurs $(i = 1, 2, \ldots$, then $S_{t_i(\alpha)} = d \inf\{m : md \geq \alpha\}$, $S_{t_i(\alpha)-1}$ is positive, and

$$\mathsf{E}\{S_{\tau_2(\alpha)} | S_{\tau_1(\alpha)}, \mathfrak{A}_2(\alpha)\} = \mathsf{E}\{S_{\tau_2(\alpha)} | \mathfrak{A}_2(\alpha)\} = \mathsf{E}\{S_{\tau_1(\alpha)} | \mathfrak{A}_1(\alpha)\} > 0.$$

In addition,

$$\mathsf{E}\{S_{\tau_1(\alpha)} | \mathfrak{A}_1(\alpha)\} = \mathsf{E}\{S_{t_1(\alpha)} | \mathfrak{A}_1(\alpha)\} - \mathsf{E}X_t^- \mathsf{P}\{X_t \leq 0\} - (-\mathsf{E}\{S_\tau | X_1 > 0\})\mathsf{P}\{X_t > 0\}$$

$$= \mathsf{E}\{S_{t_1(\alpha)} | \mathfrak{A}_1(\alpha)\} - d - (-\mathsf{E}\{S_\tau | X_1 > 0\})\mathsf{P}\{X_t > 0\}.$$

Hence, the constant b is

$$b = \mathsf{E}\{S_{t_1(\alpha)-1} | \mathfrak{A}_1(\alpha)\} - (\mathsf{E}\{S_{t_1(\alpha)} | \mathfrak{A}_1(\alpha)\} - d - (-\mathsf{E}\{S_\tau | X_1 > 0\})\mathsf{P}\{X_t > 0\})$$

$$= -d + d + \mathsf{E}\{S_\tau | X_1 > 0\}\mathsf{P}\{X_t > 0\}$$

$$= \mathsf{E}\{S_\tau | X_1 > 0\})\mathsf{P}\{X_t > 0\},$$

and we arrive at the statement of the theorem. $\qquad\square$

Corollary 4.10. *In the case where X_t^+ takes only one single positive value d, and X_t^- takes the only values $\{0, -d\}$ from Theorem 4.9 we obtain*

$$\mathsf{E}L_\alpha = \frac{1}{2}\mathsf{P}\{X_1 \neq 0\},$$

which is correct for all $\alpha > 0$.

Proof. Indeed, in this case $\mathsf{E}\{S_\tau | X_1 > 0\} = 0$. Hence, the result follows from Theorem 4.9. $\qquad\square$

Let us now discuss purely symmetric random walks. In the case of these random walks, Corollary 4.8 is simplified as follows. If Assumption 4.1 is satisfied, then $\mathsf{E}\{S_{t_1 - 1} | \mathfrak{A}_1(\alpha)\}$ and $\mathsf{E}\{S_{\tau_1} | \mathfrak{A}_1(\alpha)\}$ must be equal and we have the following statement.

Theorem 4.11. *Let S_t be a purely symmetric random walk, and let Assumption 4.1 be satisfied for some $\alpha_0 > 0$. Then, for all levels $\alpha \geq \alpha_0$,*

$$\mathsf{E}L_\alpha = \frac{1}{2} \cdot \frac{a - \mathsf{E}\{S_\tau | X_1 > 0\}}{a}\mathsf{P}\{X_1 \neq 0\}.$$

Proof. For any $\alpha > \alpha_0$, because the random walk is purely symmetric,

$$\mathsf{P}\{S_{t_i(\alpha)-1} \in B | \mathfrak{A}_i\} = \mathsf{P}\{S_{\tau_i(\alpha)} \in B | \mathfrak{A}_i\}$$

for all $i = 1, 2, \ldots$ and any Borel set $B \in \mathbb{R}^1$. Hence,

$$b_\alpha = \mathsf{E}\{S_{t_i(\alpha)-1} | \mathfrak{A}_i\} - \mathsf{E}\{S_{\tau_i(\alpha)} | \mathfrak{A}_i\} = 0.$$

Applying the arguments of the proof of Theorem 4.6 we arrive at the desired conclusion. $\qquad\square$

Remark 4.12. From Theorem 4.11 we conclude as follows. Let $\{S_t, r\}$ be a family of all purely symmetric random walks with the given probability $r = \mathsf{P}\{X_1 \neq 0\}$. Apparently, the minimum of $\mathsf{E}L_\alpha$ for the family $\{S_t, r\}$ is achieved in the case when $\mathsf{E}\{S_\tau | X_1 > 0\} = 0$, and for all $\alpha > 0$ we obtain:

$$\min_{\{S_t, r\}} \mathsf{E}L_\alpha = \frac{1}{2}r.$$

Example 4.13. Let us return to Example 3.3. For the purely symmetric random walk that specified there, it was shown $\mathsf{E}L_1 = \frac{1}{2}$. Let us evaluate $\mathsf{E}L_\alpha$, for all $\alpha \geq 2$. Notice that $\mathsf{P}\{S_1 = 1|X_1 > 0\} = \mathsf{P}\{S_1 = 2|X_1 > 0\} = \frac{1}{2}$, and

$$\mathsf{P}\{S_t = 1|1 \leq S_i \leq 2, i = 1, 2, \ldots, t\} = \mathsf{P}\{S_t = 2|1 \leq S_i \leq 2, i = 1, 2, \ldots, t\} = \frac{1}{2}.$$

Hence, for $\alpha = 2$, $\mathsf{E}\{S_{t_1(2)-1}|\mathfrak{A}_1\} = \mathsf{E}\{S_{\tau_1(2)}|\mathfrak{A}_1\} = \frac{2}{3} > 0$, and $\mathsf{E}\{S_\tau|X_1 > 0\} = -\frac{1}{3}$. Keeping in mind that $a = \frac{3}{2}$, by Theorem 4.11, for all $\alpha \geq 2$ we have

$$\mathsf{E}L_\alpha = \frac{1}{2} \cdot \frac{\frac{3}{2} + \frac{1}{3}}{\frac{3}{2}} = \frac{11}{18}.$$

5. APPLICATION TO QUEUING THEORY

In this section, the application of level-crossings in random walks is demonstrated for elementary queuing problems in a not traditional formulation.

Consider a $M^X/M^Y/1/N$ queuing system in which the expected interarrival time of some random quantity X (which we associate with an arrival of a *customer* for convenience) is equal to $\frac{1}{\lambda}$, the expected service time of some random quantity Y is equal to $\frac{1}{\mu}$. X characterizes a 'weight' of a customer (say mass), and Y characterizes a 'capacity' (say mass) of a service. The random variables X and Y are assumed to be positive random variables and not necessarily integer, and N is assumed to be a positive real number in general.

Let X_1, X_2,... denote consecutive weights of customers having the same distribution as X, and let Y_1, Y_2,... denote the service capacities having the same distribution as Y. The sequences $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ are assumed to be independent and to consist of independently and identically distributed random variables.

The *full rejection policy* is supposed. In this policy, if at the moment of arrival of a customer the capacity N of the system is exceeded, then a customer (with his/her entire weight) leaves the system without any service.

Let L_N denote the number of losses during a busy period. In the theorem below we study the expected number of losses during a busy period. For this specific queuing system, the behaviour of the number of losses during a busy period differs from that of the number of level-crossings in the associated random walk. However, the arguments that were used in the proof of Theorem 4.6 are applied here as well.

Theorem 5.1. *Assume that* $\lambda \mathsf{E}X = \mu \mathsf{E}Y$. *Then for any* $N > \mathsf{E}X$ *and any non-trivial random variable* Y *(i.e. taking at least two positive values) we have the inequality* $\mathsf{E}L_N > 1$.

Proof. Let $p = \frac{\lambda}{\lambda+\mu}$, and let $q = \frac{\mu}{\lambda+\mu}$. Denote by S_t an associated random walk in which $S_0 = 0$ and $S_1 = X_1$ and the following values are $S_t = S_{t-1} + W_t$, $t = 2, 3, \ldots$, where $W_t = X_t \mathsf{I}(A) - Y_t \mathsf{I}(\overline{A})$, A and $\overline{A}$ are opposite events. The event A occurs with the probability p and the event $\overline{A}$ occurs with the complementary probability q. Thus, the associated random walk is a E-symmetric random walk. Let $t_1, t_2, \ldots,$ t_{L_N} be the moments when the customers are lost from the system.

Then the difference between the structure of a E-symmetric random walk and the queuing process is as follows. The random variables S_{t_i-1} and S_{τ_i} have generally different expectations in relations (4.4) and (4.5), and the difference between them is denoted b. The expected number of level-crossings in Theorem 4.6 is expressed via this quantity b. In the case of the queuing system considered here, we have the equality $S_{t_i-1} = S_{\tau_i}$, where S_{t_i-1} is the value of the queuing capacity before the moment when the loss from the system occurs, and S_{τ_i} is the value of capacity after the loss. More specifically, in the case of queuing system the time moments $t_i - 1$ and τ_i are the same, and being compared with those of associated random walk they can be considered as coupled. In other words, the random walk is "cut" in the points $t_i - 1$ and they are "coupled" with the points τ_i. Then apparently, $\mathsf{E}\{S_{t_i-1}|\text{the } i\text{th loss occurs}\} = \mathsf{E}\{S_{\tau_i}|\text{the } i\text{th loss occurs}\}$, and the application of the same arguments as those in the proof of Theorem 4.6 in the given case should be made with $b = 0$. It is only taken into account that for any nontrivial random variable Y (taking at least two positive values) we have $\mathsf{E}\{S_\tau|X_1 > 0\} < 0$, where S_τ is the stopping time of the "cut" random walk as explained above. The physical meaning of the inequality $\mathsf{E}\{S_\tau(N)|X_1 > 0\} < 0$ is associated with the case that the last service batch in a busy period is incomplete.

As in the case of associated E-symmetric random walk, for the value N the inequality $N > \mathsf{E}X$ should be taken into account in order to guarantee the condition $\mathsf{E}\{S_{t_1-1}|\text{the first loss occurs}\} > 0$. Then, similarly to the main result of Theorem 4.6 we have:

$$(5.1) \qquad\qquad \mathsf{E}L_N = \frac{a - \mathsf{E}\{S_\tau|X_1 > 0\}}{a},$$

where the only difference is that S_τ is related to the "cut" random walk, and unlike in the usual random walk now in depends on N as well, i.e. $S_\tau = S_\tau(N)$. Since $\mathsf{E}\{S_\tau(N)|X_1 > 0\}$ is strictly negative for any N, from (5.1) we finally obtain $\mathsf{E}L_N > 1$. □

Remark 5.2. According to Theorem 5.1, for any non-trivial random variable Y we have $\mathsf{E}L_N > 1$. (The values $\mathsf{E}L_N$ generally depend of N, because for different N the values $\mathsf{E}\{S_\tau|X_1 > 0\}$ can be different.) In the case of the $M^X/M/1/N$ queuing system where X is a positive integer random variable and $Y = 1$ we have $\mathsf{E}L_N = 1$ for all $N \geq 0$, because in this case $\mathsf{E}\{S_\tau|X_1 > 0\} = 0$. This result, remains correct for $M^X/GI/1/N$ queuing systems with generally distributed service times (see [1], [12] and [19]). However, in the case of the $M^X/M/1/N$ queuing system where X is a positive *continuous* random variable and $Y = 1$ the equality $\mathsf{E}L_N = 1$ does not hold. In this case we have the inequality $\mathsf{E}L_N > 1$, because when X is a continuous random variable, the last service batch in a busy period is incomplete, and we have $\mathsf{E}\{S_\tau|X_1 > 0\} < 0$.

Example 5.3. Consider a very simple example of the problem where X takes discrete values 0.1 and 0.2 with the equal probability $\frac{1}{2}$ and Y takes the same values 0.1 and 0.2 with the same probability $\frac{1}{2}$ each. The equal values of λ and μ are not a matter, let they both be equal to 1. Let $N = 1$. For this specific example, the value $\mathsf{E}L_N$ can be evaluated similarly to that of Example 4.13. According to Corollary 4.11, the value $\mathsf{E}L_N$ coincides with the expected number of level-crossings in associated random walk, given that its first jump is positive. So,

$$\mathsf{E}L_N = \frac{\frac{15}{100} + \frac{1}{30}}{\frac{15}{100}} = \frac{11}{9}.$$

Since the arrival and departure processes are symmetric, $\mathsf{E}L_N$ is the same for all N as in the associated random walk.

Example 5.4. Consider the example of the above $M^X/M^Y/1/N$ queuing system where parameters λ and μ both are equal to 1, the random variable X is generally distributed with mean 0.15 and the random variable Y is exponentially distributed with the same mean 0.15 and $N = 1$. In this case $\mathsf{E}L_N$ is independent of N as well and obtained exactly. Indeed, according to the property of the lack of

memory of the exponential distribution in the associated random walk we have $\mathsf{E}\{S_\tau|X_1 > 0\} = -0.15$, and hence,

$$\mathsf{E}L_N = \frac{.15 + .15}{.15} = 2.$$

6. Concluding remarks

In the present paper we studied level-crossings of symmetric random walk. We addressed the questions formulated in Section 3. We showed that under specified conditions given by Assumption 4.1 for E-symmetric random walks there exists the increasing sequence of levels such that the expected number of level-crossings remains the same. We obtained the expected number of level-crossings for special class of E-symmetric random walks (Theorem 4.9). For purely symmetric random walks we established a more general result saying that the expected number of level-crossings remains the same for all levels that greater some initial value α_0. It follows from Theorem 4.6 and Corollary 4.8 that for E- and P- symmetric random walks the value $\mathsf{P}\{X_1 > 0\}$ is not the minimum of the expected number of level-crossings within these classes. We showed, however (see Remark 4.12), that within the class of purely symmetric random walks, the expected number of level-crossings is not smaller than $\frac{1}{2}\mathsf{P}\{X_1 \neq 0\}$, and this lower bound is within the class of these random walks. Thus we addressed the second question formulated in Section 3. Finally, we obtained non-trivial results for the expected number of losses during a busy period of loss queueing systems.

There is a number of possible directions for the future work. One of them can be associated with the case when Assumption 4.1 is not satisfied. A new study, stimulated by Example 4.5, can be provided under the following assumption.

Assumption 6.1. Assume that $\mathsf{E}\{S_{t_1(\alpha)-1}|\mathfrak{A}_1(\alpha)\} > 0$, $\mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\} > 0$, and

$$\mathsf{E}\{S_{\tau_2(\alpha)}|\mathfrak{A}_2(\alpha)\} = \mathsf{E}\{S_{\tau_1(\alpha)}|\mathfrak{A}_1(\alpha)\}.$$

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References

[1] ABRAMOV, V.M. On a property of a refusals stream. *J. Appl. Probab.* **34** (1997) 800-805.

[2] ABRAMOV, V.M. On losses in $M^X/GI/1/n$ queues. *J. Appl. Probab.* **38** (2001) 1079-1080.

[3] ABRAMOV, V.M. Takács' asymptotic theorem and its applications. A survey. *Acta Appl. Math.*, **109** (2010) 609-651.

[4] ADLER, R.J. AND HASOFER, A.M. Level-crossings for random fields. *Ann. Probab.* **4** (1976), 1-12.

[5] BOROVKOV, K.A. AND LAST, G. On level-crossings for a general class of piece-wise deterministic Markov processes. *Adv. Appl. Probab.* **40** (2008) 815-834.

[6] BOROVKOV, K.A. AND LAST, G. On Rice's formula for stationary multivariate piecewise smooth processes. ArXiv: 1009.3885v1

[7] BURRIDGE, P. AND GUERRE, E. The limit distribution of level-crossings of a random walk and a simple unit root test. *Econometric Theory*, **12** (1996), 705-723.

[8] FELLER, W. *An Introduction to Probability Theory and Its Applications*, vol. 1, 3rd edn. John Wiley, New York, 1971.

[9] FELLER, W. *An Introduction to Probability Theory and Its Applications*, vol. 2, 2nd edn. John Wiley, New York, 1968.

[10] HASOFER, A.M. The mean number of maxima above high level in Gaussian random fields. *J. Appl. Probab.* **13** (1976), 377-379.

[11] NOVIKOV, A.A. Estimates and asymptotic behavior of the probability of not crossing moving boundaries by sums of independent random variables. *Izv. Acad. Nauk SSSR, Ser. Mat.* **44** (1980), 868-885. (Russian.)

[12] RIGHTER, R. (1999). A note on losses in $M/GI/1/n$ queues. *J. Appl. Probab.*, **36**, 1240-1243.

[13] PEKÖZ, E., RIGHTER, R. AND XIA, C.H. Characterizing losses during busy periods in finite buffer systems. *J. Appl. Probab.* **40** (2003) 242-249.

[14] SHIRYAYEV, A.N. *Probability*, 2nd edn, Springer, Berlin, 1995.

[15] SIEGMUND-SCHULTZE, R. AND VON WEIZSÄCHER, H. Level-crossing probabilities I. One-dimensional random walks and symmetrization. *Adv. Math.* **208** (2007), 672-679.

[16] SPITZER, F. *Principles of Random Walk*, 2nd edn. Springer, Heidelberg, 2001.

[17] SZEKÉLY, G.J. *Paradoxes in Probability Theory and Mathematical Statistics*. Akademiai Kiado, Budapest, 1986.

[18] WOLFF, R.W. *Stochastic Modeling and the Theory of Queues*. Prentice Hall, Englewood Cliffs, 1989.

[19] WOLFF, R.W. Losses per cycle in a single-server queue. *J. Appl. Probab.* **39** (2002) 905-909.

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